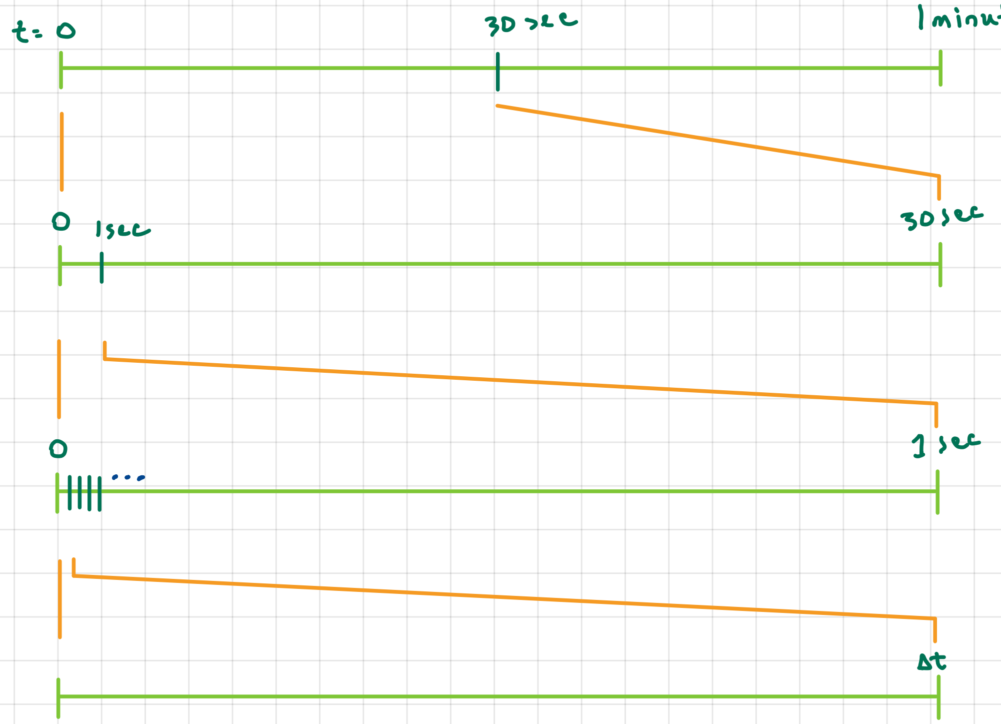
Poisson probability   counting independent arrivals.

Suppose on average 120 packets / minute



avg: 120 packets / min

avg: 60 packets / 30 sec

avg: 2 packets / sec.

avg: <<< 1 packet / Δt.

"rate" remains constant

$$\lambda = 120 \frac{1}{\text{min}} \cdot \underbrace{\left(\frac{1}{\Delta t}\right)}_{n \text{ intervals}} \cdot \Delta t$$

$\therefore$ very small $\Delta t \longrightarrow$ P[1 arrival in interval] = p. <<< 1 (≈ 0)

$\therefore$ $\underbrace{P[\text{2 arrivals}] = p^2, \quad P[\text{3 arrivals}] = p^3, \ldots}_{\text{independent arrivals}}$

$\therefore$ $\Delta t \longrightarrow 0$, P[1 arrival] small ; P[more than 1 arrival] ≈ 0.

$\therefore$ binomial structure : $\underset{\text{large}}{n}$ flips, P[1 arrival] = $\underset{\text{small}}{p}$, with $\underbrace{n \cdot p}_{\text{constant rate}} = \lambda$.

Thrm (Poisson Law) $b \xrightarrow{d} P$ if $n \gg 1$ and $p \ll 1$ and $\lambda = np$.

Pf: $$\lim_{n\to\infty} b(n,k,p) = \lim_{n\to\infty} \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \lim_{n\to\infty} \frac{n!}{k!(n-k)!} \left(\frac{\lambda}{n}\right)\left(1-\frac{\lambda}{n}\right)^{n-k}$$ if $\lambda = np$.

$$= \frac{\lambda^k}{k!} \lim_{n\to\infty} \frac{n!}{(n-k)!\,n^k} \left(1-\frac{\lambda}{n}\right)^{n}\left(1-\frac{\lambda}{n}\right)^{-k}$$

$$= \frac{\lambda^k}{k!} \lim_{n\to\infty} \frac{n\cdot(n-1)\cdots(n-k+1)}{n\quad n\quad \cdots \quad n} \left(1-\frac{\lambda}{n}\right)^{n}\left(1-\frac{\lambda}{n}\right)^{-k}$$

$$= \frac{\lambda^k}{k!} \lim_{n\to\infty} 1\cdot\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k+1}{n}\right)\left(1-\frac{\lambda}{n}\right)^{n}\left(1-\frac{\lambda}{n}\right)^{-k}$$

$$= \frac{\lambda^k}{k!} \underbrace{\lim_{n\to\infty}\left(1-\frac{1}{n}\right)}_{=1}\cdots \underbrace{\lim_{n\to\infty}\left(1-\frac{k+1}{n}\right)}_{=1}\cdot \underbrace{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n}}_{e^{-\lambda}}\cdot \underbrace{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{-k}}_{=1}$$

$$= \frac{\lambda^k}{k!} e^{-\lambda}$$

$$= P(\lambda)$$

QED.

Poisson pdf BEG CUP

issue spot: counting structure

$X \sim P(\lambda)$ for parameter ("mean") $\lambda > 0$

$$P(X=k) = \frac{e^{-\lambda}\lambda^k}{k!} \qquad k = 0, 1, 2, \ldots$$

Ex: Alpha-decay of Radium to Radon

$$^{226}_{88}Ra \longrightarrow {}^{222}_{86}Rn + {}^{4}_{2}He^{2+}$$

α-particle

half-life of ^{226}Ra isotope: 1600 years

Radium: 226 grams / mole

$\therefore$ 1 gram $\approx 10^{22}$ atoms $(= n)$

Binary event: atom "decays" spontaneously or not.

"Vast inter-atomic distance" $\longrightarrow$ independence

$\lambda =$ α-emissions per-gram per-second

$\approx 10^{10}$ decays

$T = 1$ second

$$\therefore P(1 \ \alpha\text{-decay}) = b(10^{25}, 1) \approx \frac{10^{10}}{10^{22}} = 10^{-12}$$

$$= \frac{1}{\text{trillion}}$$

$$\therefore P(X(t) = k) \approx \frac{e^{-\lambda}\lambda^{k}}{k!} \quad \text{for } k = 0, 1, 2, \ldots$$

Ex: Basketball long shots $\quad X \sim b(70, 0.02)$

assume independent shots

$$\therefore \begin{cases} n = 70 \gg 1 \\ p = 0.02 \ll 1 \end{cases}$$

$\therefore \lambda = n \cdot p = 70 \cdot (0.02) = 1.4$

$\therefore$ Exact binomial probability

$$P(3 \text{ baskets in } 70 \text{ trials}) = \binom{70}{3}(0.02)^{3}(0.98)^{67}$$

$$= 0.1131$$

$\therefore$ Poisson approximation with $\lambda = np = 1.4$

$$P(X=3) = \frac{(1.4)^3 e^{-1.4}}{3!} = 0.1128$$

"moment generating function"

Later $\quad b(n,k,p) \longleftrightarrow m_B(s) = (1-p+pe^s)^n$

$$\left(= E[e^{sX}] \right)$$

$$P(\lambda) \longleftrightarrow m_p(s) = e^{\lambda(e^s-1)}$$

$\therefore$ For $n \cdot p = \lambda$ constant.

$$\lim_{n\to\infty} m_B(s) = \lim_{n\to\infty} \left(1 - \frac{\lambda}{n} + \frac{\lambda}{n} e^s\right)^n \qquad \text{since } p = \frac{\lambda}{n}$$

$$= \lim_{n\to\infty} \left(1 - \frac{\lambda(e^s-1)}{n}\right)^n$$

$$= e^{\lambda(e^s-1)}$$

$$= m_p(s)$$

$\therefore$ Again: $b(n,k,p) \xrightarrow{d} P(\lambda)$ if $\lambda = n \cdot p$ (and Levy's Theorem week 11)

Accuracy of Poisson Approximation

Okay:	$n \geq 20$ and	$p \leq 0.05$
Excellent:	$n \geq 100$ and	$p \leq 0.1$

Measuring the size of sets

Defn: $f: X \to Y$ is 1-to-1 ("Injective") iff

$$\forall x \forall z : f(x) = f(z) \longrightarrow x = z.$$

Defn: $f: X \to Y$ is ONTO ("Surjective") iff

$$\forall y \in Y \; \exists x \in X : y = f(x) \qquad (f(X) = Y)$$

Defn: $f: X \to Y$ is a BIJECTION ("1-to-1 correspondence")

iff f is 1-to-1 and onto.

$|A|$ = cardinality ("size") of set A.

$\mathbb{N} = \{1, 2, 3, \ldots\}$ natural numbers

$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$ integers

$\mathbb{Z}^+ = \mathbb{N}$

$\mathbb{R} = (-\infty, \infty)$ reals

Defn: A is finite iff $A \xleftrightarrow[\text{onto}]{\text{1-to-1}} S$ for some $S \subset \mathbb{N}$

such that $A = \{a_1, \ldots, a_n\}$ for some $n \in \mathbb{N}$ (or $A = \emptyset$)

Else A is infinite.

Fact: A infinite $\longleftrightarrow$ $A \xleftrightarrow[\text{onto}]{\text{1-to-1}} B$ for some $B \subset A$ and $B \neq A$.

$\therefore$ infinite $\longleftrightarrow$ $\sim$finite

Cantor's theorem: $|X| < |2^X|$

Defn: A is denumerable iff $A \underset{\text{onto}}{\overset{\text{1-to-1}}{\longleftrightarrow}} \mathbb{N}$ (e.g. $\mathbb{Z}$)

Defn: A is countable iff (1) A is finite, or

(2) A is denumerable

Facts: $|\mathbb{Z}| = |\mathbb{N}| = \aleph_0$ (aleph-nought)

$|\mathbb{R}| = |2^{\mathbb{Z}}| = c = \aleph_1$

↑ power of the continuum

$|A| = \aleph_k \implies |2^A| = \aleph_{k+1}$ $(k = 0, 1, \dots)$

Cantor's Continuum Hypothesis There is no w such that

$$\aleph_k < w < \aleph_{k+1}$$

Ex: $(0,1)$ same size as $\mathbb{R}^+$

$$f(x) = \frac{1}{1+e^{-x}}$$

(note: adding 2-points $0, 1 \longrightarrow [0,1]$ no 1-to-1 onto map w/ $\mathbb{R}^+$)

Aside: $\pm\infty$ not real numbers